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Localization Stabilized by Noise

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We present a model which describes a quantum two-state system interacting with the environment represented by stochastic noise. We show that coherent tunneling between the two states survives if the interaction with the environment is weak. On the contrary, a strong interaction destroys quantum coherence and the system randomly jumps from one state to the other. Moreover, the jump probability rate becomes extremely small for very strong noise. The model is relevant for understanding the quantum properties of some mesoscopic systems.

KEY WORDS: Quantum tunneling; stochastic processes.

The distinction between microsystems described by quantum mechanics and macrosystems described by classical mechanics has been for decades a paradigm of physics. Quantum behavior for large systems is obviously not theoretically impossible, but it was excluded by the *a priori* assumption that it was practically not detectable. In recent years a mesoscopic region of physics has become accessible to experiment.[1-3] As a consequence, the problem has been revisited and the idea that a large system could have some quantum behavior has become popular. Typical and widely studied examples of these systems are superconducting rings with a "weak" junction[1,2,4] crossed by a magnetic flux. These objects, under appropriate conditions, have only two possible configurations, and coherent tunneling between them is possible provided that the interaction with the environment is not too strong. This is a crucial point. In fact, it turns out that the description of these new phenomena has to take carefully into account the interaction with the environment.

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In this paper we present an extremely simplified model which can be solved exactly. The model describes a two-state system, such as a superconducting ring, interacting with an environment represented by a kind of stochastic noise. When there is no interaction with the environment the system jumps from one state to the other periodically: this corresponds to typical quantum coherent tunneling. In this case the probability of being localized in the first or the second state oscillates periodically. We show that damped periodic oscillations of the localization probability survive if the interaction with the environment is sufficiently weak. In this situation the quantum tunneling from one state to the other is still coherent. On the contrary, a strong interaction destroys quantum coherence, and in fact the localization probability relaxes exponentially. One thus says that the system jumps from one state to the other randomly or, better, that tunneling is not coherent. The phenomenon is indistinguishable from a classical stochastic process. Moreover, we surprisingly find that the rate of the exponential relaxation of the localization probability becomes extremely small for very strong noise. This indicates that a strong interaction with the environment has a stabilizing effect on the localization: not only has the quantum tunneling disappeared, but stochastic jumps are suppressed. Numerical evidence for this kind of stabilization has been also found by Grossmann *et al.*,[7] who studied a bistable system in the framework of stochastic resonance.[8]

Our results can be qualitatively compared with those of Chakravarty and Leggett[5] and Leggett *et al.*[6] on a two-state system interacting with a dissipative environment described as a boson field. The average behavior of the localization probability in our model shows the same qualitative features. However, our model allows us to perform an explicit computation of fluctuations and correlations, which are absent in a purely dissipative system.

Let us consider the time-dependent Hamiltonian

$$H = \alpha\sigma_x + \beta\eta(t)\,\sigma_z \tag{1}$$

where σ_x and σ_z are Pauli matrices and η is a given realization of a white noise [i.e., $w(t) \equiv \int_0^t \eta(s)\,ds$ is a Brownian motion]. When $\beta = 0$, this Hamiltonian produces coherent quantum tunneling of period π/α between the two eigenstates of σ_z which have the same mean energy and represent the two "macroscopic" configurations of the system. When $\beta \neq 0$ we introduce an interaction with the environment which randomly breaks in time the energy symmetry of the two configurations.

The associated Schrödinger equation is, by Îto calculus,

$$d\Phi = -i\alpha\sigma_x\Phi\,dt - i\beta\sigma_z\Phi\,dw - \tfrac{1}{2}\beta^2\Phi\,dt \tag{2}$$

where

$$\mathbf{\Phi} = \begin{pmatrix} \Phi_+ \\ \Phi_- \end{pmatrix}$$

is a two-component vector. $|\Phi_+|^2$ is the probability of finding the system in the first macroscopic configuration, while $|\Phi_-|^2$ is that of finding it in the second one. It is easy to check that Eq. (2) corresponds to the unitary evolution $\mathbf{\Phi}(t) = U_w(t)\,\mathbf{\Phi}(0)$ associated to the Hamiltonian (1).

We are now interested in the localization properties of the model. It is therefore convenient to introduce the vector $\mathbf{x} = (x, y, z)$ defined as the mean value $\langle \mathbf{\Phi}\, \boldsymbol{\sigma}\, \mathbf{\Phi} \rangle$ of the quantum vector $\boldsymbol{\sigma}$ with respect to the quantum state $\mathbf{\Phi}$. The third component $z = |\Phi_+|^2 - |\Phi_-|^2$ of this vector encodes the information about the localization ($|\Phi_+|^2 + |\Phi_-|^2 = 1$). Starting from the Schrödinger equation (2), we find that the vector $\mathbf{x}$ satisfies

$$d\mathbf{x} = A\mathbf{x}\, dt + B\mathbf{x}\, dw \tag{3}$$

where

$$A = \begin{pmatrix} -2\beta^2 & 0 & 0 \\ 0 & -2\beta^2 & -2\alpha \\ 0 & 2\alpha & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & -2\beta & 0 \\ 2\beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The above stochastic equation is equivalent to the Schrödinger equation and can be interpreted as a diffusion on a unitary sphere with radius fixed by the initial conditions, where for us $|\langle \sigma \rangle|^2 = x^2 + y^2 + z^2 = 1$.

From (3), taking the expectation value, we immediately obtain

$$\frac{d\bar{\mathbf{x}}}{dt} = A\bar{\mathbf{x}} \tag{4}$$

This linear equation can be easily solved. When $\beta^2 < 2\,|\alpha|$ the solution is

$$\begin{aligned} \overline{x(t)} &= e^{-2\beta^2 t}\,\overline{x(0)} \\ \overline{y(t)} &= e^{-\beta^2 t}[\overline{y(0)} \cos(\omega t) + c_1 \sin(\omega t)] \\ \overline{z(t)} &= e^{-\beta^2 t}[\overline{z(0)} \cos(\omega t) + c_2 \sin(\omega t)] \end{aligned} \tag{5}$$

where

$$\omega = |\beta^4 - 4\alpha^2|^{1/2}, \qquad c_1 = [-\beta^2\overline{y(0)} - 2\alpha\overline{z(0)}]/\omega, \qquad \text{and}$$

$$c_2 = [\beta^2\overline{z(0)} + 2\alpha\overline{y(0)}]/\omega.$$

Looking at the third component $z(t)$, one realizes that in this region a quantum coherent behavior survives the noise. The localization probability is periodic with exponential damping: the system jumps from one state to the other almost periodicially. The damping factor, in fact, is a consequence of the accumulation of errors due to the deviations from the purely periodic behavior. The damping rate β^2 increases with noise.

When $\beta^2 > 2|\alpha|$ the solution is purely exponential and it may be obtained from (5) with the substitutions $\cos(\omega t) \to \cosh(\omega t)$, $\sin(\omega t) \to \sinh(\omega t)$. In this strong-noise region the coherent behavior is completely destroyed since the localization probability relaxes exponentially. Therefore, the system jumps randomly from one state to the other, with probability rate $\beta^2 - |\beta^4 - 4\alpha^2|^{1/2}$ which vanishes for very strong noise as $2\alpha^2/\beta^2$.

Summarizing, we have damped oscillations for a weak interaction with the environment ($\beta^2 < 2\,|\alpha|$) and incoherent relaxation for strong interaction ($\beta^2 > 2\,|\alpha|$). Chakravarty and Leggett[5] and Leggett *et al.*[6] consider a two-state system interacting with an environment described as a boson field. The resulting spin-boson model has a behavior which depends on the interaction properties encoded in a spectral function. While detailed comparison is not possible, since their model is much more complex then ours, it is astonishing that they find the same two qualitative behaviors: damped oscillations and incoherent relaxation.

Our model in spite of its simplicity has the advantage of allowing an evaluation of fluctuations. The above expectation values of $\mathbf{x}(t)$, in fact, are not sufficient to describe the behavior of the system. We also need to compute the fluctuations of the localization probability and the correlation function. From the stochastic equations (3) one can derive the set of linear equations

$$\begin{aligned}
\frac{d\overline{z^2}}{dt} &= 4\alpha\overline{zy} \\
\frac{d\overline{y^2}}{dt} &= -4\alpha\overline{zy} - 4\beta^2\overline{y^2} + 4\beta^2\overline{x^2} \\
\frac{d\overline{yz}}{dt} &= -2\beta^2\overline{zy} + 2\alpha\overline{y^2} - 2\alpha\overline{z^2}
\end{aligned} \tag{6}$$

Using the condition $x^2+y^2+z^2=1$, one can replace $\overline{x^2}$ with $1-\overline{y^2}-\overline{z^2}$ in the second of the above equations. For any $\beta \neq 0$ one finds that the solution converges to the stationary solution $\overline{x^2}=\overline{y^2}=\overline{z^2}=1/3$, $\overline{zy}=0$. This result says that the localization probability of the system is itself a random quantity both in the weak- and strong-interaction regions as long as $\beta \neq 0$. Even for large times there are symmetric fluctuations around $z=0$. In fact, for large times, one has $\overline{z^2}=1/3 \neq \bar{z}^2=0$.

In order to compute the correlation function

$$c(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_0^T z(s+\tau)\,z(s)\,ds \tag{7}$$

it is useful to remark that the exponential convergence to the stationary solution implies

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\int_0^T \overline{x(s)^2}\,ds &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \overline{y(s^2)}\,ds\\ &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \overline{z(s)^2}\,ds=\frac{1}{3}\end{aligned} \tag{8}$$

and

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \overline{z(s)\,y(s)}\,ds=0 \tag{9}$$

We preliminary compute the mean correlation function $\overline{c(\tau)}$. This can be done by taking first the average $\overline{z(s+\tau)}$ conditioned to the initial value $z(t)$. This is a trivial application of (5). The resulting expression is a linear combination of the integrals which appear in (8) and (9) and the result is straightforward. Using the same procedure, one shows that $[\overline{c(\tau)}]^2=\overline{c(\tau)^2}$. The quantity $c(\tau)$ is therefore nonrandom. The result for $\beta^2<2\,|\alpha|$ is

$$c(\tau)=\frac{1}{3}e^{-\beta^2\tau}\left[\cos(\omega\tau)+\frac{\beta^2}{\omega}\sin(\omega\tau)\right] \tag{10}$$

while for $\beta^2>2\,|\alpha|$ it is

$$c(\tau)=\frac{1}{3}e^{-\beta^2\tau}\left[\cosh(\omega\tau)+\frac{\beta^2}{\omega}\sinh(\omega\tau)\right] \tag{11}$$

This provides clear evidence that there are two regions. The first corresponds to a weak interaction with the environment. The system still jumps almost periodically from one state to the other so that coherent quantum behavior survives. On the contrary, in the strong-interaction region, the most typical quantum phenomenon disappears. Surprisingly, the jump probability rate vanishes for very large β so that, in the limit of extremely strong noise, the behavior of the system becomes "classical." This result leads to the important conclusion that the noise stabilizes the localization since the jump probability rate becomes $2\alpha^2/\beta^2$.

The relevance of our model for the description of the quantum behavior of some mesoscopic systems stems from the fact that the control parameter $\beta^2/|\alpha|$ can be of the order of unity for these objects. The evaluation of the effective strength of the coupling with the environment becomes crucial.

The inhibition of transition due to the coupling with the environment seems to be a typical behavior of quantum systems strongly interacting with macroscopic objects. This effect has been extensively studied in the relevant case of interaction with a macroscopic measuring device inducing consecutive wave packet collapses. The phenomenon is known as the quantum Zeno effect and it has been clearly evidenced both theoretically[12,13,16,17] and experimentally.[13,14]

The model that we have introduced seems simple and rich enough to become a test model to check the properties of two-state mesoscopic systems. It seems also to be relevant in the quantum chaos problem when a quasiperiodic forcing replaces the stochastic noise.[9,10] The chaotic properties of the model in terms of information entropy are presently under investigation.[11]

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